Complex Analysis (for Physics)
Resit Exam / Solutions
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1) Consider the complex function

$$F(x+iy) = (x^3 + 3xy^2 - 3x) + i(y^3 + 3x^2y - 3y)$$

a) Determine the point(s) at which F is differentiable.
b) Compute the derivative F' at the point(s) where it exists.

Solution. a) We start by identifying the real and imaginary parts of the function, i.e.

$$u(x,y) = \operatorname{Re} F(z) = x^3 + 3xy^2 - 3x$$

and

$$v(x,y) = \operatorname{Im} F(z) = y^3 + 3x^2y - 3y.$$

This shows that the function is defined everywhere with continuously differentiable real and imaginary parts, therefore $F(z)$ is differentiable exactly at the points $z_0 = x_0 + iy_0 \in \mathbb{C}$ where the Cauchy-Riemann equations are satisfied (cf. Theorem 5 of Section 2.4). The Cauchy-Riemann equations read

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \iff \left\{\begin{aligned} 3x^2 + 3y^2 - 3 &= 3y^2 + 3x^2 - 3 \\ 6xy &= -6xy \end{aligned}\right.$$

The first equation is an identity, i.e. it holds for all x, y whereas the second equation implies that $12xy = 0$ meaning that $x = 0$ or $y = 0$. This means that the Cauchy-Riemann equations are satisfied at points along the real and imaginary axes hence the derivative of $F(z)$ exists along these axes.

b) Given that the derivative exists at $x \in \mathbb{R}$ and $iy \in i\mathbb{R}$, we have

$$F'(x) = \frac{\partial u}{\partial x}(x,0) + i\frac{\partial v}{\partial x}(x,0), \quad F'(iy) = \frac{\partial u}{\partial x}(0,y) + i\frac{\partial v}{\partial x}(0,y).$$

Based on part a), we see that

$$F'(x) = [(3x^2 + 3y^2 - 3) + i(6xy)]\,|_{(x,y)=(x,0)} = 3x^2 - 3,$$

$$F'(iy) = [(3x^2 + 3y^2 - 3) + i(6xy)]\,|_{(x,y)=(0,y)} = 3y^2 - 3.$$

2) Consider the complex function

$$g(z) = \frac{z}{z^2 - z - 2}$$

a) Find its Taylor series and the circle of convergence around 0.
b) Find its Laurent series expansion in the domain $|z| > 2$.
c) Determine its singularities (with type and order specified).

Solution. a) Using partial fraction decomposition we find that

$$g(z) = \frac{2/3}{z-2} + \frac{1/3}{z+1}$$

which is suitable for applying the geometric series formula. Namely, the first term can be written as

$$\frac{2/3}{z-2} = -\frac{2/3}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{if } \left|\frac{z}{2}\right| < 1, \text{ i.e. } |z| < 2$$

and the second term can be expressed as

$$\frac{1/3}{z+1} = -\frac{1}{3}\frac{1}{1-(-z)} = -\frac{1}{3}\sum_{n=0}^{\infty}(-z)^n \quad \text{if } |-z|<1, \text{ i.e. } |z|<1.$$

Therefore for points inside the unit circle $|z| < 1$ we have the following Taylor series expansion of $g(z)$ around 0:

$$g(z) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}\frac{1}{3}\left((-1)^{n+1} - \frac{1}{2^n}\right)z^n.$$

b) The choice of domain $|z| > 2$ suggests a different way of expressing the terms in the partial fraction decomposition seen in part a). Namely, we have

$$\frac{2/3}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{2/3}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \quad \text{if } \left|\frac{2}{z}\right|<1, \text{ i.e. } |z|>2$$

and

$$\frac{1/3}{z+1} = \frac{1/3}{z}\frac{1}{1-\left(-\frac{1}{z}\right)} = \frac{1/3}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n \quad \text{if } \left|-\frac{1}{z}\right|<1, \text{ i.e. } |z|>1.$$

Thus the Laurent series of $g(z)$ in the domain $|z| > 2$ reads

$$g(z) = \frac{2/3}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1/3}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^{n+1}+(-1)^n}{3}z^{-(n+1)}.$$

c) Being a rational function, the singularities of $g(z)$ in $\mathbb{C}$ are poles located at the zeros of the denominator (with matching orders). Therefore the singularities of $g(z)$ are simple poles located at $z = 2$ and $z = -1$.

3) Compute the following complex integral

$$\oint_{\Gamma}(i\overline{z} - 5z)\,dz$$

where Γ is the positively oriented contour consisting of the interval $[0, 4]$ and the lower semicircle of radius 2 centered at 2.

<u>Solution.</u> The contour is the union of a line segment C_1 and a semicircle C_2 which can be parameterized by the functions $z_1(t) = 4 - t$, $0 \le t \le 4$ and $z_2(t) = 2 + 2e^{it}$, $-\pi \le t \le 0$, respectively. Note that we have $z_1'(t) = -1$ and $z_2'(t) = 2ie^{it}$. Using the additivity of complex integrals we get

$$\begin{aligned}\oint_{\Gamma}(i\overline{z} - 5z)\,dz &= \int_{C_1}(i\overline{z} - 5z)\,dz + \int_{C_2}(i\overline{z} - 5z)\,dz\\ &= \int_0^4[i(4-t) - 5(4-t)](-1)\,dt + \int_{-\pi}^0[i(2+2e^{-it}) - 5(2+2e^{it})]2ie^{it}\,dt\\ &= \int_0^4(i-5)(t-4)\,dt + \int_{-\pi}^0(-4)[1 + (1+5i)e^{it} + 5ie^{2it}]\,dt\\ &= [(i-5)(\tfrac{1}{2}t^2 - 4t)]_0^4 + [(-4)(t + \tfrac{1+5i}{i}e^{it} + \tfrac{5}{2}e^{2it})]_{-\pi}^0\\ &= -4\pi.\end{aligned}$$

4) Evaluate the following improper integral

$$\int_{-\infty}^{\infty}\frac{x\cos(x)}{x^2 - 4x + 3}\,dx$$

Solution. Let us consider the complex function

$$f(z) = \frac{ze^{iz}}{z^2 - 4z + 3}.$$

By Euler's formula, it is clear that for $x \in \mathbb{R}$ we have

$$\operatorname{Re} f(x) = \frac{x\cos(x)}{x^2 - 4x + 3}.$$

The function $f(z)$ has two simple poles at $z_1 = 1$ and $z_2 = 3$. Take $0 < \varepsilon < 1$ and $R > 3 + \varepsilon$ and let $\Gamma_{R,\varepsilon}$ denote the positively oriented closed contour consisting of the union of intervals $[-R, 1-\varepsilon] \cup [1+\varepsilon, 3-\varepsilon] \cup [3+\varepsilon, R]$ along the real axis and the upper semicircles $C_R^+(0) \cup -C_\varepsilon^+(1) \cup -C_\varepsilon^+(3)$ (orientation indicated). Note that the poles z_1, z_2 are in the exterior of the contour $\Gamma_{R,\varepsilon}$ and that $f(z)$ is regular along and inside $\Gamma_{R,\varepsilon}$. Therefore, by Cauchy's Integral Theorem, we have

$$\oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = 0,$$

that is

$$\int_{-R}^{1-\varepsilon} f(x)\,dx + \int_{1+\varepsilon}^{3-\varepsilon} f(x)\,dx + \int_{3+\varepsilon}^{R} f(x)\,dx + \int_{C_R^+(0)} f(z)\,dz - \int_{C_\varepsilon^+(1)} f(z)\,dz - \int_{C_\varepsilon^+(3)} f(z)\,dz = 0.$$

Note that we have

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0} \operatorname{Re}\left(\int_{-R}^{1-\varepsilon} f(x)\,dx + \int_{1+\varepsilon}^{3-\varepsilon} f(x)\,dx + \int_{3+\varepsilon}^{R} f(x)\,dx\right) = \int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2 - 4x + 3}\,dx,$$

whereas

$$\lim_{R\to\infty} \int_{C_R^+(0)} f(z)\,dz = 0$$

due to Jordan's lemma as we have $\deg(z^2 - 4z + 3) \geq 1 + \deg(z)$ for the rational part of $f(z)$. As for the remaining integrals, we have semicircular contours centred at *simple* poles. Therefore the Laurent series of $f(z)$ around these poles has only one term with negative z exponent, i.e. around $z = 1$ we have

$$f(z) = \frac{\operatorname{Res}(f,1)}{z-1} + \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n$$

and around $z = 3$ we have

$$f(z) = \frac{\operatorname{Res}(f,3)}{z-3} + \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(z-3)^n$$

The regular (Taylor series) parts are bounded around the poles hence their integrals tend to 0 as the radius $\varepsilon \to 0$. To evaluate the integrals of the singular parts we need to compute the residues. We have

$$\operatorname{Res}(f,1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\left(\frac{ze^{iz}}{z-3}\right) = \frac{e^i}{-2}$$

and

$$\operatorname{Res}(f,3) = \lim_{z\to 1}(z-3)f(z) = \lim_{z\to 3}\left(\frac{ze^{iz}}{z-1}\right) = \frac{3e^{3i}}{2}.$$

Thus parametrizing $C_\varepsilon^+(1)$ via $z(t) = 1 + \varepsilon e^{it}$, $0 \leq t \leq \pi$ we get

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon^+(1)} f(z)\,dz = \lim_{\varepsilon\to 0}\int_{C_\varepsilon^+(1)} \frac{e^i}{-2}\frac{1}{z-1}\,dz = \lim_{\varepsilon\to 0}\int_0^\pi \frac{e^i}{-2}\frac{1}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt = \lim_{\varepsilon\to 0}\int_0^\pi \frac{e^i}{-2} i\,dt = -\frac{i\pi e^i}{2}.$$

In a similar way, we can parametrize $C_\varepsilon^+(3)$ via $z(t) = 3 + \varepsilon e^{it}$, $0 \le t \le \pi$ to obtain

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon^+(3)} f(z)\,dz = \lim_{\varepsilon\to 0} \int_{C_\varepsilon^+(3)} \frac{3e^{3i}}{2}\frac{1}{z-3}\,dz = \lim_{\varepsilon\to 0} \int_0^\pi \frac{3e^{3i}}{2}\frac{1}{\varepsilon e^{it}} i\varepsilon e^{it}\,dt = \lim_{\varepsilon\to 0}\int_0^\pi \frac{3e^{3i}}{2} i\,dt = \frac{3i\pi e^{3i}}{2}.$$

By combining our results for the contour integrals Cauchy's Theorem takes the following concrete form

$$\int_{-\infty}^{\infty} f(x)\,dx + \frac{i\pi e^i}{2} - \frac{3i\pi e^{3i}}{2} = 0$$

as $R \to \infty$ and $\varepsilon \to 0$. Rearranging this relation and taking the real part yields the integral in question

$$\int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2 - 4x + 3}\,dx = \frac{\pi}{2}(\sin 1 + 3\sin 3).$$

5) Show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros in the unit disc $|z| < 1$.

Solution. For the entire functions $f(z) = 4z^2 - 1$ and $h(z) = z^6$ we have

$$|f(z)| > |4z^2| - 1 = 4|z|^2 - 1 = 3 > 1 = |z|^6 = |z^6| = |h(z)|$$

along the unit circle $|z| = 1$. (Here we used the reverse triangle inequality and the multiplicative property of the complex modulus.) Therefore, by Rouché's Theorem, $f(z) + h(z) = z^6 + 4z^2 - 1$ and $f(z) = 4z^2 - 1$ have the same number of zeros inside the unit disc. The function $f(z) = 4z^2 - 1$ clearly has exactly two simple zeros at $z = \pm 1/2$ and they are both in the unit disc, therefore $f(z) + h(z) = z^6 + 4z^2 - 1$ also has exactly two zeros satisfying $|z| < 1$.